

An Extended Linear MPC for Nonlinear Processes

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Abstract— Nonlinear behavior is a common feature of all real-world systems. However, for the sake of simplicity, a linear model is often used in the controller design procedure. Nevertheless, the neglected nonlinear dynamics could degrade the performance of controller drastically. This study presents a new method of designing a model predictive controller (MPC) for a class of nonlinear systems. In the proposed method, an MPC is first designed in state space based on a linear model and then modified by using modal series to compensate for the effect of the neglected nonlinear dynamics in the linear model. Because the proposed controller adjusts a linear controller instead of designing a new one, it can be easily applied in industries to modify controllers that have been designed based on linear models. In addition, its computational burden is much less than that of nonlinear MPC methods. In this study, the proposed technique is used to control two real-world systems, and the results of its application are discussed.

Index Terms—Modal series, Nonlinear dynamics, Predictive control.

I. INTRODUCTION

BECAUSE of the high performance and simplicity, model predictive control (MPC) have extended their application in various industries [1-3]. To apply MPC methods in control a plant, an appropriate model of the plant, which is called a predictive model, should first be acquired. This model should be capable of predicting the system behavior to provide the designer with the required outputs in predictive horizon k by using the system information up to moment t . In mathematical terms, it should be able to predictive $\hat{y}(t+k|t)$. The predictive model and other MPC elements, such as the cost function or optimization method, determine the control rule. To decrease the computational burden, it is desirable to find an explicit control rule, which also helps the designer to determine the effective parameters on the system behavior.

Real-world systems have a nonlinear behavior; however, in many cases, their behavior can be estimated around an operating point by a linear model. As there is no limitation on the model used in MPC, it can be applied to both linear and nonlinear models. Although the use of nonlinear models leads to reasonable accuracy, an explicit and closed-form answer to the optimization problem usually cannot be determined. Most nonlinear MPC approaches use iterative and numerical procedures to solve the optimization problem. It not only results in a high numerical burden, but also the effective

parameters on the system's behavior cannot be determined obviously. On the other hand, the use of linear models usually leads the optimization problem to a closed-form solution. The required computational capacity to implement the controller is decreased, and the effective parameters on the system behavior can be recognized clearly.

According to the above discussion, in many cases in which the process can be estimated by good approximation around an operating point, the controller is designed based on a linear model and then applied to the real system. However, as getting far from the operating point, the validity of the linear model decreases, and the neglected dynamics in the linear model result in degrading the controller performance. Various methods have been proposed to compensate for the effects of the neglected dynamics; some of them are briefly reviewed in the following section.

Extended dynamic matrix control (EDMC) is a modified DMC method that can be used to control nonlinear processes [4]. In this method, the nonlinear model of a process is assumed to be available. However, to calculate the control rule, an instantaneous linear model of the process is used, which is updated at each sampling time.

EDMC requires an extra optimization problem to minimize the difference between the instantaneous linear model and the nonlinear model as much as possible. This optimization problem is solved iteratively by numerical methods. As a result, EDMC requires a higher calculation capacity compared with regular DMC [5]. However, it is less complicated than nonlinear MPC and has a significantly decreased calculation burden.

Robust MPC (RMPC) methods are another option to compensate for the neglected nonlinear dynamics. The main drawback of RMPC is its conservative nature [6-9]. In designing an RMPC, the nonlinear continuous-time model is first converted into a discrete-time model, which is then separated into two parts: linear and nonlinear terms. The nonlinear part is treated as an uncertainty in the design procedure.

The present study deals with the neglected nonlinear dynamics by using modal series, which is a new method of studying nonlinear dynamics in state space. Modal series decomposes a nonlinear system into a set of linear subsystems. As a result, many linear system concepts can be extended to nonlinear ones. The method was first proposed for the analysis of large-scale nonlinear systems, such as power systems [10-11]. It is so-called because the method

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calculates the system response based on the system natural modes. In [11], modal series is extended to the analysis of discrete-time-varying systems. Depending on the application, modified forms of modal series can be used. For instance, in [12], the Kronecker product was used to simplify the calculations. Extended and modified forms of modal series have been used in electronics, control, and other fields [13-17]. In [13], a new form was proposed for the analysis of an oscillator transient response in the time domain, whereas in [14], the main idea was to use extended modal series to solve nonlinear boundary value problems (BVPs). Further, [15] applied extended modal series to solve a class of infinite-horizon nonlinear optimal control problems (OCPs). The proposed method avoided directly solving nonlinear two-point boundary value problems (TPBVP) and Hamilton-Jacobi-Bellman (HJB) equations. In [16], a new method for solving nonlinear optimization problems with a second-order performance index was suggested. In [17], a new off-line nonlinear MPC system for continuous-time input-affine nonlinear systems was suggested.

In the present work, an MPC is first designed in state space based on a linear model and then modified through an extended form of modal series such that the effects of the neglected nonlinear dynamics in the linear model are compensated for.

Compared with EDMC, the proposed method requires less computational effort and can be applied to stable and unstable systems. On the other hand, compared with robust MPC methods, the suggested technique is less conservative because it is not designed based on the worst case. It also works faster because it avoids solving time-consuming LMI optimization problems.

Moreover, because the proposed controller adjusts a linear controller instead of designing a new one, it can be easily applied in industries to modify controllers that have been designed based on linear models.

The rest of this study is organized as follows. In Section 2, the main research idea and a new form of modal series that is used to compensate for the effects of neglected nonlinear dynamics are introduced. In Section 3, the proposed method is applied to two case study systems, and the results of its application are discussed. Finally, section 4 presents the conclusions.

II. MPC MODIFICATION THROUGH MODAL SERIES

A. Model predictive control

Consider the following nonlinear system

$$\dot{x}(t) = F(x(t), u(t)) \quad , \quad x(0) = x_0 \quad (1)$$

$$y(t) = C_m x(t) \quad (2)$$

where $x(t)$ is an n-dimensional state vector, $u(t)$ is an m-dimensional input vector, and $F: R^N \times R^M \rightarrow R^N$ is an analytic function.

In MPC applications, the model is usually converted into a discrete-time one as follows:

$$x(k+1) = H(x(k), u(k)) \quad , \quad x(0) = x_0 \quad (3)$$

$$y(k) = C_m x(k) \quad (4)$$

Suppose that the process can be approximated around an operating point (x_{op}, u_{op}) by the following linear model:

$$x_m(k+1) = A_m x_m(k) + B_m u_m(k) \quad , \quad x(0) = x_0 \quad (5)$$

$$y_m(k) = C_m x_m(k) \quad (6)$$

where $x_m(k) \approx x(k) - x_{op}$ and $u_m(k) \approx u(k) - u_{op}$.

Remark 1. Without loss of generality, hereinafter, the origin is assumed to be the operating point, i.e., $F(0,0) = 0$.

Remark 2. Model (5-6) is an approximation around the origin. This approximation is valid as long as the states and input signals are near it. By getting far away, the validity of the linear model decreases, and the neglected dynamics in the linear model result in a degraded performance. The aim of the current study is to compensate for these effects by using modal series.

Further, assume that the linear model (5-6) is used as the predictive model in designing an MPC. As previously stated, the use of this simple model instead of the nonlinear one results in a closed-form solution that not only reduces the computational burden of the algorithm but also provides the designer with effective parameters in the response. However, for the sake of performance, the cost function J is defined based on the nonlinear model as:

$$J = (R_s - Y_{nl})^T (R_s - Y_{nl}) + \Delta u^T \bar{R} \Delta u \quad (7)$$

where $\Delta u = u(k+1) - u(k)$ and Y_{nl} shows the predicted outputs of the nonlinear model defined as follows:

$$Y_{nl} = \begin{bmatrix} y(k_i + 1|k_i) \\ y(k_i + 2|k_i) \\ \vdots \\ y(k_i + N_p|k_i) \end{bmatrix} = \begin{bmatrix} C_m x(k_i + 1|k_i) \\ C_m x(k_i + 2|k_i) \\ \vdots \\ C_m x(k_i + N_p|k_i) \end{bmatrix} \quad (8)$$

Also, R_s in (7) denotes a vector consisting of the desired operating points. Because only the regulation problem is studied in this work, we set $R_s^T = [0 \ 0 \ 0 \ \dots \ 0]^T$; \bar{R} is a positive definite diagonal matrix defined as $\bar{R} = r_\omega I_{N_c \times N_c}$ ($r_\omega \geq 0$), where r_ω is a parameter used for performance adjustment.

According to (7), Y_{nl} is required to minimize the cost function. However, because the linear model (5-6) is used as the predictive model, only Y_{lin} can be obtained, which is defined as:

$$Y_{lin} = \begin{bmatrix} y_m(k_i + 1|k_i) \\ y_m(k_i + 2|k_i) \\ \vdots \\ y_m(k_i + N_p|k_i) \end{bmatrix} = \begin{bmatrix} C_m x_m(k_i + 1|k_i) \\ C_m x_m(k_i + 2|k_i) \\ \vdots \\ C_m x_m(k_i + N_p|k_i) \end{bmatrix} \quad (9)$$

It is evident that there is a difference between Y_{lin} and Y_{nl} ; therefore, Δy is defined as:

$$\Delta y = Y_{nl} - Y_{lin} \quad (10)$$

In the following, a theorem and two corollaries are derived to calculate Δy analytically.

Remark 3. A similar idea is used in EDMC, in which vector Δy is obtained by solving a nonlinear equation with the use of iterative numerical methods; in the proposed method, Δy is calculated analytically, thus increasing the speed of the algorithm. In addition, similarly to DMC, EDMC is applicable only to open loop stable processes, whereas the proposed method can be applied to unstable processes as well.

B. Model predictive control

Theorem. The response of the nonlinear system (1) can be expressed as:

$$x(t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x_{ij}(t) \quad (11)$$

Where

$$\begin{cases} x_{00}(t) = 0 \\ \dot{x}_{10}(t) = A x_{10}(t) \quad , \quad x_{10}(0) = x_0 \\ \dot{x}_{ij}(t) = A x_{ij}(t) + B_{ij} u_{ij}(t) \\ x_{ij}(0) = 0 \quad \text{for: } i, j \geq 0 \ \& \ (i, j) \notin (0,0), (1,0) \end{cases} \quad (12)$$

and A , B_{ij} , and $u_{ij}(t)$ are as defined in [18].

Proof. See [18].

Figure 1 presents the idea of the theorem. Equation (11) shows that the response of the nonlinear system (1) can be categorized into three classes: $x_{i0}(t)$, $x_{0j}(t)$, and $x_{ij}(t)$.

- $x_{i0}(t)$ is the response to the initial conditions and is the zero input response of the system.
- $x_{0j}(t)$ is the response to the input signals and is the zero state response of the system.
- $x_{ij}(t)$, where $i, j \neq 0$, is due to the interaction between the system input signals and the initial conditions.

now define

$$\begin{aligned} Y_{01} &= [C_m x_{01}(k_i + 1|k_i) \quad \dots \quad C_m x_{01}(k_i + N_p|k_i)]^T \\ Y_{10} &= [C_m x_{10}(k_i + 1|k_i) \quad \dots \quad C_m x_{10}(k_i + N_p|k_i)]^T \\ Y_{02} &= [C_m x_{02}(k_i + 1|k_i) \quad \dots \quad C_m x_{02}(k_i + N_p|k_i)]^T \\ Y_{20} &= [C_m x_{20}(k_i + 1|k_i) \quad \dots \quad C_m x_{20}(k_i + N_p|k_i)]^T \\ Y_{11} &= [C_m x_{11}(k_i + 1|k_i) \quad \dots \quad C_m x_{11}(k_i + N_p|k_i)]^T \\ &\vdots \end{aligned} \quad (13)$$

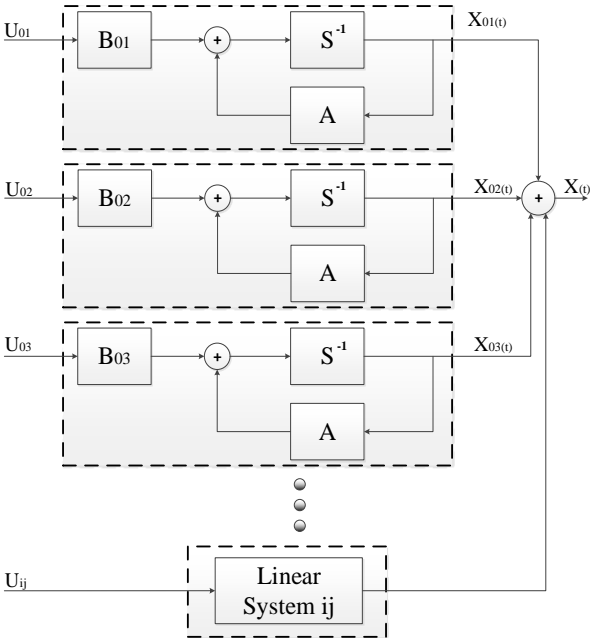


Fig. 1. Modal series concept.

Corollary 1.

$$Y_{lin} = Y_{01} + Y_{10} \quad (14)$$

Proof.

According to linear system theory, the response of a linear system consists of its response to its initial conditions and to the input signals. Considering (12), x_{10} is shown to be the response of the linear model to the initial conditions. Also considering $B_{01} = B = \frac{\partial f(x,u)}{\partial u} \Big|_{x=x_{op}, u=u_{op}}$ in (12) [18], x_{01} is the response of the linear model to the input signals. Consequently, we have

$$x_m = x_{10} + x_{01} \quad (15)$$

Now, by using (9), (13-1), (13-2), and (15), we conclude that $Y_{lin} = Y_{01} + Y_{10}$.

Corollary 2.

$$\Delta y = Y_{nl} - Y_{lin} = Y_{02} + Y_{20} + Y_{11} + \dots \quad (16)$$

Proof. By using the theorem, (8) can be expressed as:

$$Y_{nl} = C_m \begin{bmatrix} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x_{ij}(k_i + 1|k_i) \\ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x_{ij}(k_i + 2|k_i) \\ \vdots \\ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x_{ij}(k_i + N_p|k_i) \end{bmatrix} \quad (17)$$

According to the definitions in (13), (17) can be rewritten as:

$$Y_{nl} = Y_{01} + Y_{10} + Y_{02} + Y_{20} + Y_{11} + \dots \quad (18)$$

Combining (14) and (18), we conclude that the difference between the outputs of linear and nonlinear models can be expressed as:

$$\Delta y = Y_{nl} - Y_{lin} = Y_{02} + Y_{20} + Y_{11} + \dots$$

This completes the proof.

C. MPC modification through the proposed corollary

In this subsection, an MPC is first designed in state space based on a linear model, and then the proposed corollary is used to compensate for the effects of the neglected nonlinear dynamics in the linear model. Considering the following definitions:

$$\Delta x_m(k+1) = x_m(k+1) - x_m(k)$$

$$\Delta u(k) = u_m(k) - u_m(k-1)$$

$$x_f(k) = [\Delta x_m(k+1) \quad y(k)]^T$$

$$\Delta U = [\Delta u(k_i) \quad \Delta u(k_i+1) \quad \dots \quad \Delta u(k_i+2)]^T$$

we derive the augmented state space model as:

$$\begin{bmatrix} x_f(k+1) \\ \Delta x_m(k+1) \\ y_m(k+1) \end{bmatrix} = \begin{bmatrix} A_a & & \\ & A_m & 0_m^T \\ & C_m A_m & 1 \end{bmatrix} \begin{bmatrix} x_f(k) \\ \Delta x_m(k) \\ y_m(k) \end{bmatrix} + \begin{bmatrix} B_a \\ B_m \\ C_m B_m \end{bmatrix} \Delta u(k) \quad (19)$$

$$y_m(k) = \begin{bmatrix} c_a \\ 0 \quad I \end{bmatrix} \begin{bmatrix} \Delta x_m(k) \\ y(k) \end{bmatrix} \quad (20)$$

Based on the above state-space model (A_a, B_a, C_a) , the predicted output variables can be expressed as:

$$y_m(k_i + 1|k_i) = C_a A_a x_f(k_i) + C_a B_a \Delta u(k_i) \quad (21)$$

$$y_m(k_i + 2|k_i) = C_a A_a^2 x_f(k_i) + C_a A_a B_a \Delta u(k_i) + C_a B_a \Delta u(k_i + 1)$$

$$\begin{aligned} y_m(k_i + N_p|k_i) &= C_a A_a^{N_p} x_f(k_i) \\ &\quad + C_a A_a^{N_p-1} B_a \Delta u(k_i) \\ &\quad + C_a A_a^{N_p-2} B_a \Delta u(k_i + 1) + \dots \\ &\quad + C_a A_a^{N_p-N_c} B_a \Delta u(k_i + N_c - 1) \end{aligned} \quad (23)$$

Now, it is easy to show

$$Y_{lin} = Fx(k_i) + \Phi \Delta U \quad (24)$$

where

$$F = \begin{bmatrix} CA \\ CA^2 \\ CA^3 \\ \vdots \\ CA^{N_p} \end{bmatrix}$$

$$\Phi = \begin{bmatrix} CB & 0 & 0 & \dots & 0 \\ CAB & CB & 0 & \dots & 0 \\ CA^2 B & CAB & CB & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ CA^{N_p-1} B & CA^{N_p-2} B & CA^{N_p-3} B & \dots & CA^{N_p-N_c} B \end{bmatrix}$$

To find the optimal Δu , the cost function (7) is used. Because the purpose of control is regulation, the cost function is modified into the following form:

$$J = (Y_{lin} + \Delta y)^T (Y_{lin} + \Delta y) + \Delta U^T \bar{R} \Delta U \quad (25)$$

By using (24), the cost function J can be expressed as:

$$\begin{aligned} J &= (Fx(k_i) + \Delta y)^T (Fx(k_i) + \Delta y) \\ &\quad - 2\Delta U^T \Phi^T (Fx(k_i) + \Delta y) \\ &\quad + \Delta U^T (\Phi^T \Phi + \bar{R}) \Delta U \end{aligned} \quad (26)$$

Differentiating the cost function J in (26) with respect to ΔU and then setting it equal to zero, we obtain

$$\frac{\partial J}{\partial \Delta U} = -2\Phi^T(Fx(k_i) + \Delta y) + 2(\Phi^T\Phi + \bar{R})\Delta U = 0 \quad (27)$$

Assuming that $(\Phi^T\Phi + \bar{R})^{-1}$ exists, the optimized control signal is obtained as:

$$\Delta U = (\Phi^T\Phi + \bar{R})^{-1}\Phi^T(Fx(k_i) + \Delta y) \quad (28)$$

In (28), Δy shows the difference between the responses of linear and nonlinear models and can be calculated by using Corollary 2.

III. CASE STUDIES

In this section, the proposed method is applied to two case study models, and the results of its application are compared with those of linear, EDMC, and RMPC (LMI) controllers to evaluate its capability to compensate for nonlinear effects. LMI-based optimization problems can be solved by the polynomial method and can be implemented online. In this method, the control signal is separated into two ingredients: feedback and feed-forward. The LMI method requires low computational effort.

Example 1 – Consider a DC/AC converter with the following state-space model [19].

$$\begin{aligned} \dot{x}_1(t) &= -\frac{x_2^2(t)}{x_1(t)} - 5x_1(t) - 5u(t) \\ \dot{x}_2(t) &= -\frac{x_2^3(t)}{x_1^2(t)} - 7x_1(t) + \left(5\frac{x_2(t)}{x_1(t)} + 2x_1(t)\right)u(t) \\ y(t) &= x_2(t) \end{aligned}$$

The selected sampling time is equal to 0.001 min. Figure 2 shows the simulation results obtained with the proposed method, assuming the operating point $x_{ss} = [0.8234 \ 1]$ and initial conditions of $x(0) = [0.1 \ 0]^T$. In the closed loop system simulation, the following parameters are used.

$$N_p = 20, \quad N_c = 3, \quad \alpha = 1, \quad \gamma = 0.0025 \\ \beta = 0.01, \quad \delta = 0.01$$

For comparison, Fig. 2 presents the responses of the linear, EDMC, and RMPC controllers. As shown in the figure, the proposed method improves the response of the linear controller and compensates for the neglected nonlinear dynamics. Compared with EDMC, the proposed method has a faster response but requires a relatively larger control signal. The settling time is about 0.02 min for the suggested method, compared with about 0.03 min EDMC. In addition, the proposed method reduces the necessary time for EDMC method calculation by 37.9%.

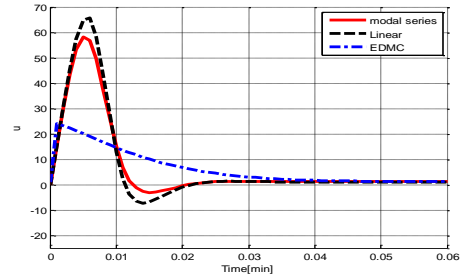
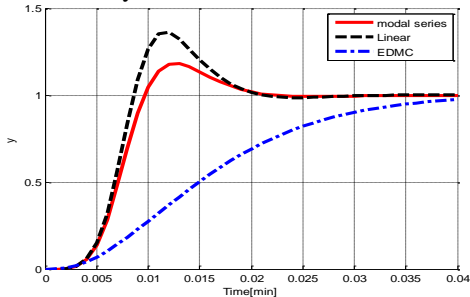


Fig. 2. Comparison between the proposed method, linear, EDMC and RMPC controllers for a DC/AC converter.

Example 2 – In this example, the proposed method is applied to a model of an isothermal reaction in an SRT, with the following state-space model [20]:

$$\begin{aligned} \dot{x}_1(t) &= -50x_1(t) - 10x_1^2(t) + (10 - x_1(t))u(t) \\ \dot{x}_2(t) &= 50x_1(t) - 100x_2(t) - x_2(t)u(t) \\ y(t) &= x_2(t) \end{aligned}$$

The selection sampling time is 0.002h. Figure 3 shows the simulation results obtained with the proposed method, assuming the operating point $y_{ss} = 1.117$ and initial conditions of $x(0) = [2 \ 1]^T$. In the closed loop system simulation, the following parameters are used:

$$N_p = 20, \quad N_c = 3, \quad \alpha = 0, \quad \gamma = 0.00069 \\ \beta = 0.01, \quad \delta = 0.01$$

For comparison, Fig. 3 plots the responses of the linear, EDMC controllers. Again, the suggested method is shown to enhance the response of the linear controller and to properly compensate for the neglected nonlinear dynamics. Although its performance is comparable with EDMC, its control signal is smoother. In addition, the proposed method reduces the necessary time for EDMC method calculation by 57.5%.

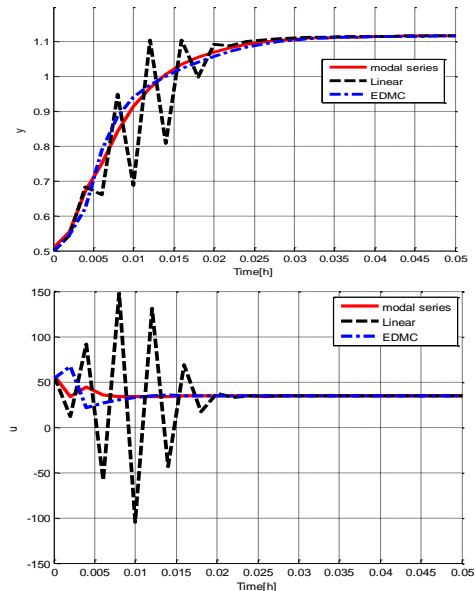


Fig. 3. Comparison between the proposed method, linear, EDMC and RMPC controllers for an isothermal reaction in an SRT.

IV. CONCLUSION

The behavior of real-world systems can be estimated around an operating point by using linear models with good accuracy. However, by getting far away from the operating point, the linear estimation accuracy decreases, and the neglected dynamics in the linear model leads to a degraded controller performance. In the present study, an MPC is first designed in state space based on a linear model and then

adjusted by using modal series to compensate for the effects of the neglected nonlinear dynamics in the linear model.

Compared with EDMC and RMPC, the proposed method requires a lower computational capacity and, unlike EDMC, can be applied to stable and unstable systems. Compared with robust MPC methods, the suggested technique is less conservative. Moreover, because the proposed controller adjusts a linear controller instead of designing a new one, it can easily be applied in industries to modify controllers that have been designed based on linear models. In this work, the proposed method is used to control a DC/AC converter and an isothermal reaction in an STR. The results show that the proposed method properly compensates for the effect of the neglected nonlinear dynamics and that its performance is comparable with that of EDMC and RMPC. In addition, the method has a significantly reduced computational burden.

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