Numerical Solution of the Advection-Dispersion Equation: Application to the Agricultural Drainage

C. Chávez, C. Fuentes∗, F. Brambila, and A. Castañeda

ABSTRACT

Subsurface drainage systems are used to control the depth of the water table and to reduce or prevent soil salinity. Water flow in these systems is described by the Boussinesq Equation, and the Advection-Dispersion Equation coupled with the Boussinesq Equation is used to study the solute transport. The objective of this study was to propose a finite difference solution of the Advection-Dispersion Equation using a linear radiation condition in the drains. The equations' parameters were estimated from a methodology based on the granulometric curve and inverse problems. The algorithm needs the water flow values, which were calculated with the Boussinesq Equation, where a fractal radiation condition and variable drainable porosity were applied. To evaluate the solution descriptive capacity, a laboratory drainage experiment was used. In the experiment, the pH, temperature, and electric conductivity of drainage water were measured to find the salt's concentration. The salts concentration evolution was reproduced using the finite difference solution of the Advection-Dispersion Equation, and the dispersivity parameter was found by inverse modelling. The numerical solution was used to simulate the leaching of saline soil. The result showed that this solution could be used as a new tool for the design of agricultural drainage systems, enabling the optimal development of crops according to their water needs and the degree of tolerance to salinity.

Keywords: Boussinesq Equation, Dispersivity parameter, Finite difference, Fractal radiation condition, Inverse modeling.

INTRODUCTION

Soil salinity is a worldwide problem as well as in Central and Northern Mexico. Nearly 8.4 million ha worldwide are affected by soil salinity and alkalinity, of which about 5.5 million ha are waterlogged (Ritzema et al., 2008). The problem worsens in arid and semiarid areas, in soils with insufficient drainage (Mousavi et al., 2009) and high evaporation (Ruiz Cerda et al., 2007). In Mexico, there are 6.46 million ha irrigated mainly in the central and northern areas (CONAGUA, 2010); 10-30% of irrigated land is affected by salinity and nearly two thirds of this area is located in the North (IMTA, 1998).

The salinization of these irrigated areas is an increasing problem and the lands are abandoned; therefore, a technical and economic alternative to recover this land is needed. Agricultural subsurface drainage is a solution which takes into account the technology by environment interaction, as well as lowering the water table levels along with the salt concentration in the soil profile (Ritzema et al., 2008).

The dynamics of water drainage systems has been studied by applying the Boussinesq Equation (1904) for unconfined aquifers using the finite element technique (Verhoest et al., 2002; Zavala et al., 2007) and the finite difference (Sing et al., 2009; Chavez et al.,...
and the solutes dynamics has been studied by applying the Fick’s law (Taylor, 1954; Elder, 1959; Fischer, 1967). These results in the Advection-Dispersion Equation, namely, by gravity and Fick’s law. The solutes are also found in the gas phase and adsorbed by soil in the solid phase, the first phase is disregarded for purposes of transport modeling in water, but it is really important in terms of the amount of fertilizer transferred into the atmosphere at a given time (Holly, 1975, 1985; Rutherford, 1994), and incorporating the adsorbed substance in the solid phase. The relationship between the substance which is transported by the water flow and the substance which is adsorbed and exchanges in the soil solid structure is known as the adsorption isotherm (Taylor, 1954; Elder, 1959; Fischer, 1967).

A large number of models for simulating solute transport in the unsaturated zone are now increasingly being used for a wide range of applications in both research and management (Mirabzadeh and Mohammadi, 2006), some of the more popular models include SWAP (van Dam et al., 1997), HYDRUS-1D (Simunek et al., 1998), STANMOD (STudio of ANalytical MODEls) (Simunek et al., 1999), UNSATH (Fayer, 2000) and COUP (Jansson and Karlberg, 2001), but the majority of applications for water flow in the vadose zone requires a numerical solution of the Richards Equation (1931), also requires more calculation time in order to find the equation solution.

This study aimed to solve the one-dimensional Advection-Dispersion Equation using the technique of finite differences, coupled with the Boussinesq Equation in order to model the transport of solutes in subsurface drainage systems, assuming that the solute is concentrated in the liquid phase.

**MATERIALS AND METHODS**

**Boussinesq Equation**

In the study of the water dynamics in agricultural subsurface drainage systems using the Boussinesq Equation, the variations in hydraulic head along the drain pipes (y direction) are negligible with respect to head variations in the cross section (x direction). It is the one-dimensional Boussinesq Equation which is a result of the Continuity Equation, $\frac{\partial (\rho H)}{\partial t} + \frac{\partial (\rho q)}{\partial x} = R_w$, and the Darcy’s law, $q = -K_s \frac{\partial H}{\partial x}$, namely:

$$\mu(H) \frac{\partial H}{\partial t} = \frac{\partial}{\partial x} \left[ T(H) \frac{\partial H}{\partial x} \right] + R_w \tag{1}$$

Where, $\mu(H)$ is the storage capacity, $H = H(x,t)$ is the elevations of the free surface or hydraulic head above the impervious layer $[L]$, and is a function of the horizontal coordinate ($x$) and the time ($t$), $T(H)$ is the transmissivity given by $T(H) = K_s H \left[ L^2 T^{-1} \right]$, $R_w$ is the volume of recharge in the unit of time per unit of the aquifer $[L^3]$, $\nu = \nu(H)$ is the drainable porosity as a head function, and $K_s$ is the saturated hydraulic conductivity $[LT^{-1}]$.

The storage capacity (Fuentes et al., 2009) is: $\mu(H) = \theta_s - \theta(H - H_s)$, where $\theta_s$ is the saturated volumetric water content $[L^3]$, and $\theta(H - H_s)$ represents the water content evolution in the position $z = H$, while the free surface decreases, and $z$ is the elevation of ground surface $[L]$.

**Drainable Porosity**

To calculate the storage capacity and the drainable porosity, it is necessary to provide the soil water retention curve. The model of van Genuchten (1980) was accepted in field and laboratory studies:

$$\theta(\psi) = \theta_s + (\theta_r - \theta_s) \left[ 1 + (\psi/\psi_d)^n \right]^{-m},$$

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where $\psi$ is the soil water potential defined by $\psi = (H - z) \ [L]$, $\psi_0$ is the pressure scale parameter $\ [L]$, $\theta_s$ is the saturated volumetric water content $\ [L^3]$, $\theta_r$ is the residual volumetric content $\ [L^3]$, $m$ and $n$ are parameters (dimensionless) that determine the shape of the soil water retention curve. The introduction of this equation in the storage capacity results in the following expression for storage capacity:

$$\mu(H) = (\theta_s - \theta_r) \left\{ 1 - \left[ 1 + \left( \frac{(H_s - H)}{|\psi_r|} \right)^m \right]^{-n} \right\}$$

The saturated volumetric water content can be assimilated to the soil porosity ($\phi$), dimensionless, this is calculated with the formula

$$\phi = 1 - \rho_b / \rho_o ,$$

where $\rho_b$ is the bulk density $\ [ML^{-3}]$ and $\rho_o$ is the particles density $\ [ML^{-3}]$; the residual volumetric water content ($\theta_r$) is considered to be zero.

**Initial and Boundary Conditions**

To study the agricultural drainage by Equation (1), the initial and boundary conditions should be defined at the domain. The initial condition is established from the water table position at the initial time. Dirichlet and Neumann boundary type conditions can be used on drains to solve Equation (1), the pressure head on the drains is required in the first condition, whereas the drainage flux is required in the second one (Zavala et al., 2007). A third type of boundary condition is a linear combination of the precedent conditions; this condition includes a resistance parameter to the flow at the soil-drain interface. Null resistance corresponds to the Dirichlet condition and infinite resistance corresponds to Neumann condition. The third condition is a radiation type condition (Carslaw and Jaeger, 1959). In the case of drainage, the radiation condition establishes that drainage flux is directly proportional to the pressure head on the drain and inversely proportional to the resistance in the interface between soil and the drainpipe wall in concordance to the Ohm’s law.

The hydraulic head measured above the impermeable barrier $H(x,t)$ is associated with the head $h(x,t)$ measured from above the drains using: $H(x,t) = D_o + h(x,t)$, where $D_o$ is the distance from the impermeable barrier to the drains $\ [L]$. Transversal variation of $h$ at the beginning is considered as the initial condition $h(x,0) = h_s(x)$, where $h_s$ is the head on the drain in the initial time $\ [L]$.

The fractal radiation condition for the Boussinesq Equations is given by Zavala et al. (2007):

$$-K_s \frac{\partial h}{\partial x} + q_s \left( \frac{h}{h_s} \right)^{2s} = 0 ; \quad x = 0, L \tag{2}$$

Where, the positive sign corresponds to $x = 0$ and the negative sign to $x = L$. $L$ is the distance between drains; $q_s$ is the corresponding flux to $h_s$ and it is a function of the soil-drain interface characteristic $\ [LT^{-1}]$. For the $s$ parameter, the authors argued that it is defined by $s = D/E$, where $D$ is the effective fractal dimension to the soil-drain interface, and $E = 3$ is the Euclidean dimension of physical space. The relation of the $s$ parameter and effective porosity is obtained from the equation

$$\left( 1 - \phi \right)^s + \phi^{2s} = 1$$

given by Fuentes et al. (2001). Equation (2) contains as particular cases the lineal radiation condition when $s = 1/2$ and the quadratic radiation condition when $s = 1$. In a system of parallel drains, the drained water flows by length unit at each drain and is given.
by:

\[ Q_d(t) = 2\left[ D_a + h(0,t)\right]q_t \left[ h(0,t)/h_1 \right]^2, \]

and the cumulative drained depth is calculated by \( \ell(t) = \frac{1}{L} \int_0^L Q_d(\tau) d\tau \), where \( \ell \) is the integration variable.

Solute Transport Equation

The Advection-Dispersion Equation used to study the solute transport (Abassi et al., 2003; Zerihun et al., 2005; Simunek, 2005) in a one-dimensional form is a result of the Continuity Equation,

\[ \partial \left( HC_s t Q_s x R \right) / \partial t + \partial Q_s / \partial x = R_s, \]

and the dynamic law given by

\[ \partial \left( HC_s t Q_s x R \right)/ \partial t + \partial \left( HqC t Q_s x R \right)/ \partial x = \partial \left( vHDA \partial C/ \partial x \right) + R_s \]

(3)

Where, \( Da \) is the diffusion coefficient in the water \( [LT^{-1}] \); \( C_r \) is the total solute concentration in soil \( [ML^{-3}] \); \( C \) is the solute concentration in water \( [ML^{-3}] \); and \( R_s \) is the term which includes gains or losses of the solute due to chemical reactions and the extraction plant \( [M] \). Note that \( q \) and \( v \) are obtained from the water flow model. The diffusion coefficient in the water is calculated by \( Da = \lambda v \), where \( \lambda \) is the dispersivity \( [L] \) and \( v \) the interstitial velocity of water calculated by \( v = q/\nu [LT^{-1}] \).

The water soluble compounds that have a negligible vapor pressure can exist in three phases in soil: (1) dissolved in water, (2) as vapor in the soil atmosphere, and (3) as stationary phase adsorbed to soil organic matter or in the clay mineral surfaces (Taylor, 1954; Elder, 1959; Fischer, 1967). The total concentration of the compound \( (C_r) \), expressed in units of mass per volume of soil can be written as: \( C_r = vC + \rho_a C_a \), where \( C_a \) is the concentration of the adsorbed compound \( [ML^{-3}] \) and is a function of the concentration of the solute in the mobile phase \( (C_d) [ML^{-3}] \) and the adsorption constant of the solute to the stationary phase surface \( (\kappa) \), \( C_a = \kappa C_d \), namely, linear isotherm. Thus, the concentration of the substance compared to the volume of the porous medium \( (C_T) \) will be the result of a part that is in the water, air and the dynamic equilibrium with the phase that generates it. Generally, in studies in small time scales, such as irrigation and drainage in a porous medium, the gas phase is not considered (Zerihun et al., 2005). Thus, in this work, the concentration in the adsorbed and in the gas phase and the term \( R_s \) are ignored.

Numerical Scheme

The numerical scheme presented is based on the assumption that the solute is concentrated mainly in the liquid phase. Thus, the Advection-Dispersion Equation in one-dimensional form is given by Equation (3). To solve this equation, we used the same discretization scheme to transfer water in the Boussinesq Equation (Chávez et al., 2011), for which two interpolation parameters are introduced:

\[ \gamma = (x_{i+\omega} - x_i)/x_{i+1} - x_i \]

and \( \omega = (t_{j+\omega} - t_j)/t_{j+1} - t_j \), where \( 0 \leq \gamma \leq 1 \) and \( 0 \leq \omega \leq 1; \ i = 1,2,... \) and \( j = 1,2,... \) are the space and time indices, respectively.

The dependent variable \( (\Phi) \) in an intermediate node \( i + \gamma \) for all \( j \) is estimated as:

\[ \Phi_{i+\gamma} = (1 - \gamma) \Phi_i + \gamma \Phi_{i+1} \]

(4)
while the intermediate time \( j + \omega \) for all \( i \) is estimated as:
\[
\Phi^{j+\omega}_i = (1 - \omega) \Phi^{i}_i + \omega \Phi^{i+1}_i
\]  

(5)

The discretization of the temporal derivative in the Equation (3) is:
\[
\frac{\partial (vHC)^{j+\omega}_i}{\partial t} = \left( vH \right)^{i+1}_i C^{j+1}_i - \left( vH \right)^{i}_i C^{j}_i + \left( \rho_i H \right)^{i+1}_i C^{j+1}_{di} - \left( \rho_i H \right)^{i}_i C^{j}_{di} = b_2 C^{j+1}_i - b_1 C^{j}_i + b_0
\]
\[
\Delta t = t_{j+1} - t_j
\]  

(6)

Where,
\[
b_0 = \frac{(\rho_i H)^{i+1}_i C^{j+1}_{di} - (\rho_i H)^{i}_i C^{j}_i}{\Delta t};
\]
\[
b_1 = \frac{(vH)^{j}_i C^{j}_i}{\Delta t};
\]
\[
b_2 = \frac{(vH)^{j+1}_i}{\Delta t}
\]  

(7)

The spatial derivative discretization in the continuity equation is:
\[
\frac{\partial Q_s^{j+\omega}_{i+\gamma}}{\partial x_i} = Q_s^{j+\omega}_{i+\gamma} - Q_s^{j+\omega}_{i-\gamma};
\]
\[
\Delta x_i = (1 - \gamma)(x_i - x_{i-1}) + \gamma(x_{i+1} - x_i)
\]  

(8)

According to the dynamic law:
\[
Q_s^{j+\omega}_{i+\gamma} = (Hq)^{j+\omega}_{i+\gamma} C^{j+\omega}_{i+\gamma} - (vH)^{j+\omega}_{i+\gamma} (Da)^{j+\omega}_{i+\gamma} \frac{C^{j+\omega}_i - C^{j+\omega}_{i+1}}{x_{i+1} - x_i}
\]  

(9)

\[
Q_s^{j+\omega}_{i-\gamma} = (Hq)^{j+\omega}_{i-\gamma} C^{j+\omega}_{i-\gamma} - (vH)^{j+\omega}_{i-\gamma} (Da)^{j+\omega}_{i-\gamma} \frac{C^{j+\omega}_i - C^{j+\omega}_{i-1}}{x_i - x_{i-1}}
\]  

(10)

According to the Equation (4), the spatial interpolation is:
\[
C^{j+\omega}_i = (1 - \gamma) C^{j}_i + \gamma C^{j+1}_i; \quad C^{j+\omega}_{i+\gamma} = (1 - \gamma) C^{j}_i + \gamma C^{j+1}_i
\]  

(11)

and according with the Equation (5) the temporal interpolation is
\[
C^{j+\omega}_i = (1 - \omega) C^{j}_i + \omega C^{j+1}_i
\]  

The dependent variables involved in the advective term of the Equations (9) and (10) are defined by:
\[
C^{j+\omega}_{i+\gamma} = (1 - \omega) C^{j+\omega}_i + \omega C^{j+\omega}_{i+1} = (1 - \omega)[(1 - \gamma) C^{j}_i + \gamma C^{j+1}_i] + \omega \left[ (1 - \gamma) C^{j+1}_i + \gamma C^{j+1}_{i+1} \right]
\]  

(12)

\[
C^{j}_{i-\gamma} = (1 - \omega) C^{j}_i + \omega C^{j}_{i-1} = (1 - \omega)[(1 - \gamma) C^{j}_i + \gamma C^{j+1}_i] + \omega \left[ (1 - \gamma) C^{j+1}_i + \gamma C^{j+1}_{i-1} \right]
\]  

(13)

while the dependent variables involved in the dispersive term of the same equations are defined by:
\[
C^{j+\omega}_{i+1} = (1 - \omega) C^{j+\omega}_i + \omega C^{j+\omega}_{i+1} = (1 - \omega) C^{j}_i + \omega C^{j+1}_i; \quad C^{j+\omega}_{i-1} = (1 - \omega) C^{j}_i + \omega C^{j+1}_i
\]  

(14)

Considering Equations (9) and (10), Equation (8) can be written as:
\[
\frac{\partial Q_s^{j+\omega}_i}{\partial x_i} = a_1 C^{j+\omega}_{i+\gamma} - a_2 \left( C^{j+\omega}_{i+1} - C^{j+\omega}_i \right) - a_3 \left( C^{j+\omega}_i - C^{j+\omega}_{i-1} \right) + a_4 \left( C^{j+\omega}_i - C^{j+\omega}_{i-1} \right)
\]  

(15)

Where,
\[
a_1 = \frac{(Hq)^{j+\omega}_{i+\gamma}}{\Delta x_i}; \quad a_2 = \frac{(vH)^{j+\omega}_{i+\gamma} (Da)^{j+\omega}_{i+\gamma}}{\Delta x_i(x_i - x_{i+1})}; \quad a_3 = \frac{(Hq)^{j+\omega}_{i-\gamma}}{\Delta x_i}; \quad a_4 = \frac{(vH)^{j+\omega}_{i-\gamma} (Da)^{j+\omega}_{i-\gamma}}{\Delta x_i(x_i - x_{i-1})}
\]  

(16)
Substituting Equations (12)-(14) in Equation (15) and associating similar terms allows obtaining:

\[
\frac{\partial Q_s}{\partial x} = -\omega \left[ a_4 + (1 - \gamma) a_5 \right] C_{i+1}^{j+1} + \omega \left[ (1 - \gamma) a_1 + a_2 - \gamma a_3 + a_4 \right] C_i^{j+1} + \omega [\gamma a_1 - a_2] C_{i+1}^{j+1} \\
-(1 - \omega) \left[ a_4 + (1 - \gamma) a_5 \right] C_{i+1}^j + (1 - \omega) \left[ a_3 - \gamma a_5 + a_2 + (1 - \gamma) a_1 \right] C_i^j \\
+(1 - \omega) [\gamma a_i - a_2] C_{i+1}^j
\]

(17)

Substituting Equations (6) and (17) in the Continuity Equation, the following algebraic equations system is obtained:

\[ A_s C_i^{j+1} + B_s C_i^j + D_s C_{i+1}^j = E_s ; \quad i = 2, 3, ..., n - 1 \]  

(18)

Where,

\[ A_s = -\omega \left[ a_4 + (1 - \gamma) a_5 \right] \]

(19)

\[ B_s = \omega \left[ (1 - \gamma) a_1 + a_2 - \gamma a_3 + a_4 \right] + b_2 \]

(20)

\[ D_s = \omega [\gamma a_i - a_2] \]

(21)

\[ E_s = R_s + (1 - \omega) \left[ a_4 + (1 - \gamma) a_5 \right] C_i^j \\
-\left\{ (1 - \omega) \left[ a_3 - \gamma a_5 + a_2 + (1 - \gamma) a_1 \right] - b_1 \right\} C_i^j - (1 - \omega) [\gamma a_i - a_2] C_{i+1}^j - b_0 \]

(22)

The water flow and the head are obtained from the Boussinesq Equation solution, so that they should be included in the system (18). To find the solution of the water transfer equation, it is necessary to specify the initial and boundary conditions, Equation (18) can be solved with the Thomas Algorithm (see Zataráin et al., 1998, Chávez et al., 2011).

The Thomas algorithm, also known as the tridiagonal matrix algorithm (TDMA), is a simplified form of Gaussian elimination that can be used to solve tridiagonal matrix systems [Equation (18)] (Freund and Hoppe, 2007). It is based on LU decomposition in which the matrix system \( Mx = r \), where \( L \) is a lower triangular matrix and \( U \) is an upper triangular matrix. The system can be efficiently solved by setting \( Ux = p \) and then solving first \( Lp = r \) for \( p \) and then \( Ux = p \) for \( x \). The Thomas algorithm consists of two steps. In the first step, decomposing the matrix into \( M = LU \) and solving \( Lp = r \) are accomplished in a single downwards sweep, taking us straight from \( Mx = r \) to \( Ux = p \). In the second step, the equation \( Ux = p \) is solved for \( x \) in an upwards sweep (Conte and De Boor, 1980).

**Linear Radiation Condition**

The radiation boundary condition, or mixed condition, is used to accept a linear variation between the dispersive flux and concentration difference with the external medium \( C_{ext} \) and the border, for all time. The linear radiation condition is due originally to Newton, who postulated that the heat flow at the border of a body is proportional to the temperature difference between the body and the medium that surrounds it; the result is equivalent to Ohm’s law in electricity. To linearize these conditions, we introduce a generalization of
the dimensionless conductance coefficient \( \kappa_s \), as follows:

\[-(\partial C/\partial x) + \kappa_s (C - C_{eq}/L) = 0.\]

If we observe the one-dimensional equation of solute transport, the dimensionless conductance coefficient \( \kappa_s \) must be zero by the advective component, however, the solution is allowed only for purposes of illustration to derive the boundary conditions.

**Selection of the Space \((\Delta x)\) and Time \((\Delta t)\) Increments**

Chavez et al. (2011) discuss the selection of spatial and temporal increments pointing out a comparison of the depletion of the free surface for all time between the results obtained with the finite difference solution of the Boussinesq Equation and the results obtained with an analytical solution reported in the literature. Chávez et al. (2011) concluded that the optimal interpolation that minimizes the sum of the squares errors are \( \gamma = 0.5\Delta x \) (cm) and \( \omega = 0.98\Delta t \) (h), for space and time, respectively.

**Laboratory Experiment**

To evaluate the descriptive capacity of the numerical solution, a drainage experiment was conducted in a laboratory. The drainage module was the one used by Zavala et al. (2007) and Chávez et al. (2011). The module dimensions were: \( L = 100\) cm, \( H_s = 120\) cm and \( D_o = 25\) cm. The drain diameter was \( d = 5\) cm and the drain length was \( l = 30\) cm. The module was filled with altered sample of salty soil of Celaya, Guanajuato, México. Soil was passed through a 2 mm sieve and was disposed on 5 cm thick layers, in order to maintain the bulk density at a constant value. The soil was saturated by applying a constant water head (no salt) on its surface until the entrapped air was virtually removed. Once the drains were closed, the water head was removed from the soil surface; the surface of the module was then covered with a plastic in order to avoid evaporation. Finally, the drains were opened to measure the drained water volume; the initial condition was equivalent to \( h(x,0) = h_o \) and the recharge was null \( R_w = 0 \) during the drainage phase. Soil porosity \( \phi \) was calculated with the formula \( \phi = 1 - \rho_s/\rho_o \) (the bulk density was determined by the weight and volume of the soil of drainage module \( \rho_s = 1.14\) g/cm\(^3\) and the particles density \( \rho_o = 2.65\) g/cm\(^3\), \( \phi = 0.5695\) cm\(^3\)/cm\(^3\) was obtained). The soil fractal dimension obtained was equal to 0.7026.

**Analysis of the Salt Content**

During the module drainage process (154 hours), measurements of pH, temperature, and electrical conductivity of water samples were made at defined time intervals (each hour during the first 20 hours and, subsequently, increased to the range 2, 4, 6 and 8 hours). The sensor used for measurement was a CONDUCTRONIC PC 18 sensor. The electrical conductivity at room temperature was recorded with it. However, in order to accurately quantify conductivity, it is important to consider a standard value of 25°C, which can be used to correct the values obtained. The correction factor used in accordance with Villareal and Bello (1964) was 2-3% for every Celsius degree that was measured under standard temperature. According to Villareal and Bello (1964), the relationship between electrical conductivity and concentration is:

\[ C = 640 \times EC \]  
(23)
Where, \( C \) is the concentration given in mg l\(^{-1}\) and \( EC \) the electrical conductivity given in dS m\(^{-1}\) or mmhos m\(^{-1}\).

**Hydrodynamic Characteristic**

To solve the Boussinesq Equation, the van Genuchten model (1980) for the water retention curve was used, along with a model of hydraulic conductivity of Fuentes et al. (2001), namely, geometric mean model

\[
K(\Theta) = K_s \left[ 1 - \left( \frac{\Theta}{\Theta_m} \right)^{sm} \right]^{2}
\]

with the restriction \( 0 < sm = 1 - 2s/n < 1 \); where \( \Theta \) is the effective saturation defined by

\[
\Theta = \left( \theta - \theta_i \right) / \left( \theta_s - \theta_i \right).
\]

**Granulometric Curve**

The \( m \) and \( n \) form parameters from the water retention curve were obtained from the granulometric curve (Fuentes, 1992) adjusted with the equation

\[
F(D) = \left[ 1 + \left( \frac{D_s}{D} \right)^N \right]^{-M},
\]

where \( F(D) \) is the cumulative frequency, based on the weight of the particles whose diameters are less than or equal to \( D \); \( D_s \) is a characteristic parameter of particle size, \( M \) and \( N \) are two form empirical parameters. These parameters are rewritten as follows: \( M = m \) and \( N = \left[ 1/2 \left( 1 - s \right) \right] n \).

**Inverse Problem**

To evaluate the capacity of the numerical solution of the Advection-Dispersion Equation, the experimental information presented by Chávez (2010) was used. The characteristics of the drainage module and the soil parameters used in the simulation were: \( h_i = 120 \) cm, \( D_0 = 25 \) cm, \( L = 100 \) cm, \( \phi = 0.5695 \) cm\(^3\) cm\(^{-3}\), and \( s = 0.7026 \). The hydrodynamic characteristics used were those of van Genuchten (1980) and Fuentes et al. (2001). The scale parameters \((\psi_d, K_s)\) were obtained from the inverse problem, using the experimental drained depth and the drained depth calculated with the numerical solution of the Boussinesq Equation (Chávez et al., 2011), given an error criterion between the previous and the new estimator \((1 \times 10^{12} \text{ cm})\), using a constant head test and fractal radiation condition with variable storage capacity and a nonlinear optimization algorithm (Marquardt, 1963). The calculations were performed on a dual-core AMD Opteron machine with 2.6 GHz CPU and 8 GB RAM. The computational time required to solve the inverse problem was 5 h.

In order to model the salt concentration in the soil profile, with the numerical solution of the solute transport, the hydraulic parameters obtained from the previous analysis were used. In the numerical solution, the unknown parameter is the dispersivity coefficient \((\lambda)\), which is estimated by minimizing the sum of squares errors between the salt concentration measured and the salt concentration calculated with the numerical solution over time, using a Levenberg-Marquardt algorithm (Marquardt, 1963), given an error criterion between the previous and the new estimator \((1 \times 10^{-9} \text{ g l}^{-1})\). The initial condition is the sample initial, taken as a constant in all the system and radiation as the boundary condition applied in the drains.

**RESULTS AND DISCUSSION**

**Granulometric Curve**

The adjusted parameters are shown in Table 1. Figure 1-A shows the experimental granulometric curve and best fit is obtained with \( D_s = 36.2993 \) µm and \( m = 0.3410 \).
Table 1. Values of the adjusted parameters from the granulometric curve and the drained depth.

<table>
<thead>
<tr>
<th>Model</th>
<th>$K_s$ (cm h$^{-1}$)</th>
<th>$\psi_d$ (cm)</th>
<th>$\kappa$ (Non-dimensional)</th>
<th>RMSE (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometric mean model</td>
<td>1.5458</td>
<td>143.87</td>
<td>0.0616</td>
<td>0.2195</td>
</tr>
</tbody>
</table>

Hydrodynamic Characteristic

In order to obtain the values of $\psi_d$ and $K_s$, the spatial and temporal increments used in all the simulation are $\Delta z = 0.0010$ cm and $\Delta t = 5 \times 10^{-5}$ h. Figure 1B shows the experimental drained depth and the drained depth calculated with the finite difference solution (Chavez et al., 2011), using a storage capacity variable, fractal radiation condition in the drains, and the geometric mean model. To linearize the boundary condition, one generalization of the conductance coefficient is optimized ($\kappa$) (Zavala et al., 2007, Chávez et al., 2011). The residual volumetric water content is considered to be zero ($\theta_r = 0.0$ cm$^3$ cm$^{-3}$) (Haverkamp et al., 2005).

Analysis of the Salt Content

The EC data are shown in Figure 1-A using a 2.5% like correction factor. Applying Equation (23) to the data shown in Figure 2-B, we obtained the concentration in grams per liter (see Figure 2-B). The initial condition used in the numerical solution is the sample initial ($C_{ini} = 2.4$ g l$^{-1}$), taken as a constant in all the system and radiation as the boundary condition applied in the drains. The dispersivity value obtained is $\lambda = 91.80$ cm, with $RMSE = 0.1063$ g l$^{-1}$ between the experimental values and the values obtained from the numerical solution. The computational time required to solve the advection-dispersion model was 2.7 hours. The dispersivity value found was only for this soil, because this value changes with depth (Simunek and van Genuchten, 1999), increases with the flow rate, and is a soil type function. This increase was explained by the activation of large pores at higher flow rates (Feyen et al., 1998). Figure 2-B shows the experimental salt concentration evolution and the concentration obtained with the numerical solution.

Comparison shows that the salt concentration obtained with the numerical solution, according to $RMSE$, reproduce the experimental salt concentration. Figure 3-A shows that in the short time, when the water...
flow increased, the salt concentration increases sharply, and in the long time, it tends toward an asymptote, indicating that the system could not continue removing salts from the system. However, the value of the dispersivity obtained ($\lambda = 91.80 \text{ cm}$) overestimates the measured data in the long time. Second simulation was performed with the accumulated mass. To obtain the accumulated mass, it was necessary to obtain the solute flow, which was estimated by multiplying the water flow by the measured salt concentration in the time interval (Figure 3-A). The cumulative solute mass was obtained by multiplying the solute flow by the time interval (Figure 3-B).

The results obtained with the numerical solution, the solute flow, and the cumulative mass evolution are shown in Figures 3-A and -B, respectively, which demonstrate that the reproductions of the data were acceptable. The solute flow decreased rapidly, as seen in Figure 3-A, the concentration decreased 3.5 g l$^{-1}$ after 20 hours. In the long time, the theoretical water flow and experimental water flow tended to be constant. Comparison showed that the solute flow and the cumulative mass evolution obtained with the numerical solution, according to $RMSE$, reproduced the experimental salt concentration. The $RMSE$ values for estimating the solute flow and cumulative solute mass were 0.1842 g l$^{-1}$ and 0.1104 g, respectively. The dispersivity value obtained was $\lambda = 98.03 \text{ cm}$, with $RMSE = 0.1010 \text{ g}$ between the experimental values and the values obtained from the numerical solution. The dispersivity value for this new optimization (cumulative mass evolution) compared to the previous (salt concentration evolution) increased 6.2 cm.

Using the Solution to Simulate the Leaching of Saline Soils

To reclaim saline soils, it is necessary to apply irrigation so that the salts are
transported to deeper horizons without harming the roots and are carried to other areas through the drainage channel. For purposes of illustrating the leaching of salts in the soil by applying the finite difference solution, we assumed a soil with hydraulic and hydrodynamic characteristics previously found. The initial soil concentration was 10 dS m\(^{-1}\) and the problem was reduced to finding the number of irrigation that must be applied to carry a given concentration.

The final average concentration obtained in the profile at the end of the first simulation was the initial concentration in the system for the next simulation, and so on. Figure 4-A shows the reduced concentration of salts in the soil profile based on an initial concentration. The values shown are an average concentration in the soil profile at 1 m depth. Depth of drains was assumed to be 2.0 m.

The simulations were performed with 5, 10, 15, 20, and 25 m of drains spacing. It can be seen that the decrease in the concentration of salts in the soil profile is similar in all the spacing between the drains after applying 6 leaching. However, the time of drainage in each system was different. For example, with 5 days and 5 m spacing, decrease of the water table profile was more than one meter, while in the system with spacing of 25 m, the decrease was only a few centimeters (see Figure 4-B); therefore, the time of drainage of the soil was a function of the distance between drains.

**CONCLUSIONS**

Irrigation in the arid and semi-arid regions to sustain agricultural production against the unpredictability of the rainfall has resulted in the added problem of salinity in many hectares of good agricultural land. Subsurface drainage systems are used to control the depth of the water table and to reduce or prevent soil salinity.

The Advection-Dispersion Equation was solved in order to model the temporal evolution of the concentration of salts removed through an agricultural drainage system with the method of finite differences. The solution requires the values of the flow of water previously obtained from the solution of the Boussinesq Equation. The hydrodynamic characteristics were obtained by the inverse problem from the depth drained.

The optimization of the accumulated mass gave better results in terms of mean square error criterion between the theoretical and experimental values, since it is a property integrated in the time and concentration observed at specific levels. The solution presented, coupled to the Boussinesq Equation, satisfactorily reproduced the measured data, both in the short time where the change in concentration was high, and in

![Figure 4](https://example.com/figure4.png)

**Figure 4.** (A) Evolution of the salt concentration in the soil by applying the leaching, (B) Decrease of the midpoint water table at different spacing between drains under a drain depth of 2.00 m.
the long times where the concentration values tended toward an asymptote. This asymptotic value of the concentration depended on the distance between drains of the drainage system.

Finally, the solution of differential equations of transfer processes of water and solute transport, and hydrodynamic characterization of the soil in an agricultural drainage system, will be a useful tool for designing new systems for the optimal growth of crops according to their water needs and degree of tolerance to salinity.

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چکیده

سامانه‌های زیرزمینی زهکشی برای کنترل عمق سقفه آب و کاهش یا جلوگیری از شوری اراضی به کار می‌روند. جریان آب در این سامانه‌ها با معادله بوسینسکی بین می‌شود و از معادله همرفتی-پراکندگی همراه با معادله بوسینسکی برای مطالعه انتقال مواد حلال شده استفاده می‌شود. هدف این پژوهش ارائه حل معادله همرفتی-پراکندگی به روش اختلال محدود (finite difference) استفاده از شرایط خطي شعاعی در زهکش‌ها بود.

پارامترهای مدل‌برداری از روش مبتنی بر منحنی دانی به دست آمده خاک و مصالح معمول بر آورد شد. الگوریتم مجزه‌بر به مقادیر جریان آب نیاز دارد که با استفاده از معادله بوسینسکی و با شرایط تشکیل فرآیندهای مختلف بودن منافع و قابلیت زهکشی محاسبه شد. برای ارزیابی طرفیت نسبی محلول، یک آزمون زهکشی در آزمایشگاه انجام شد که در آن اسیدهای نترسی، درجه حرارت، و هدایت الکتریکی شرط آب انتزاعی گیری شده با غلتک نمک تعیین شود. از سوی دیگر، تغییرات تکاملی غلتک نمک ها با استفاده از حل معادله همرفتی-پراکندگی به روش اختلال محدود و تعیین پارامتر پراکندگی با مدل سازی معموس به دست آمد. از روش حل عددی برای شبیه سازی فرایند شستشوی نمک یک خاک شروع استفاده شد. نتایج نشان داد که می‌توان از این روش به عنوان ابزار جدید در طراحی سامانه‌های زهکشی در کشاورزی و در تیم‌های فرآیند شستشوی لازم برای رشد بهبود گیری متناسب با نامی، تیز آبی آنها و درجه تحلیل‌سازی به شوری بهره‌برد.